Then the wave equation simplifies to

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

which is the same as

$$\frac{\partial^2 (rp)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2} = 0$$

This is the wave equation in rp !

So, we know that $rp = f(r-ct) + g(r+ct)$

added in Session 13

$$\boxed{p = \underbrace{\frac{1}{r} f(r-ct)}_{\text{outgoing}} + \underbrace{\frac{1}{r} g(r+ct)}_{\text{incoming}} \qquad \frac{1}{r} \equiv \text{Geometric spreading}}$$

We'll consider harmonic spherical waves; complex exponential representation;

$$\hat{p}(r,t) = \frac{\hat{A}}{r} e^{j(\omega t - kr)}$$

- outgoing wave
- ω = frequency
- $k = \omega/c$ = wavenumber

Interpret as

$$p(r,t) = \mathrm{Re}\left[\hat{p}(r,t)\right] = \frac{|\hat{A}|}{r} \cos\left(\omega t - kr + \arg \frac{\hat{A}}{r}\right)$$

$\left|\frac{\hat{A}}{r}\right| = \frac{|\hat{A}|}{r} \sim$ Pressure amplitude $\sim Pa$

$\arg \frac{\hat{A}}{r} = \arg \hat{A}$

$\hat{A}$ has units of $Pa \cdot m$

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Example ; $\hat{A} = 1 + 1j$ Pa·m , $r = 2m$

$|\hat{A}| = \sqrt{2}$; $\arg \hat{A} = 45°$

$p(r,t) = \frac{\sqrt{2}}{2} \cos(\omega t - k \cdot 2 + 45°)$ Pa

$\uparrow$ $\frac{|\hat{A}|}{r}$ is the pressure amplitude....

Computation of acoustic velocity for harmonic ; spherical waves

Given $\Rightarrow$ $\hat{p}(r,t) = \frac{\hat{A}}{r} e^{j(\omega t - kr)}$

Find $\Rightarrow$ $\hat{u}(r,t) = \hat{U} e^{j(\omega t - kr)}$: i.e., find $\hat{U}$

Use the ~~conservation~~ linearized conservation of momentum equation;

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p \; ; \quad \text{• one-D } r \text{ motion} \quad \text{• } \nabla \text{ in spherical coords.} \quad ; \; \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r}$$

Substituting $\hat{p}, \hat{u}$;

$$\rho_0 j\omega \hat{U} e^{j(\omega t - kr)} = -\frac{\partial}{\partial r}\left[\frac{\hat{A}}{r} e^{j(\omega t - kr)}\right]$$

.... after algebra....

$$\hat{U} = \frac{1}{\rho_0 c}\frac{\hat{A}}{r}\left[1 - \frac{j}{kr}\right] \Rightarrow \hat{u}(r,t) = \frac{1}{\rho_0 c}\frac{\hat{A}}{r}\left[1 - \frac{j}{kr}\right] e^{j(\omega t - kr)}$$

In the limit $kr \gg 1$

$$\hat{u}(r,t) \approx \frac{1}{\rho_0 c}\frac{\hat{A}}{r}e^{j(\omega t - kr)}$$

Same as for a rightward travelling PW with 1/r spreading

Simplifying $kr \gg 1$; $k = \frac{2\pi}{\lambda}$; $\Rightarrow$ $\boxed{r \gg 0.159\lambda}$

Criteria for spherical wave behaves like a plane wave

Numerical examples with acoustic velocity:

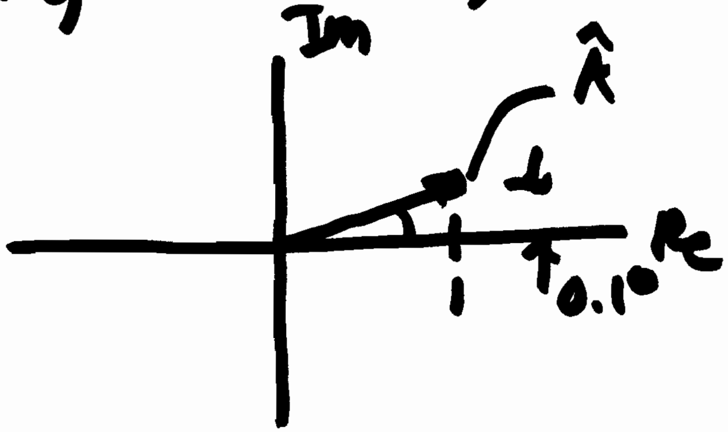
Given $\hat{A} = 1 + 0.1j$ Pa·m, $r = 3$m, $f = 2000$Hz, $\rho_0 = 1.21$, $c = 343$

Acoustic pressure and velocity amplitude = ??

$|\hat{A}| = 1.005$ Pa·m

$$\frac{|\hat{A}|}{r} = \frac{1.005\ \mathrm{Pa\cdot m}}{3} = 0.335\ \mathrm{Pa}.$$

$$\arg \hat{A} = 5.7^\circ \quad ; \quad k = \frac{2\pi f}{c} = 36.64\ /\mathrm{m}$$



$$p(3,t) = 0.335 \cos(2\pi \cdot 2000t - 36.64 \cdot 3 + 5.7^\circ)\ \mathrm{Pa}$$

$$\hat{u} = \frac{\hat{A}/r}{\rho_0 c}\left[1 - \frac{j}{kr}\right] = \frac{(1+0.1j)/3}{1.21 \cdot 343}\left[1 - \frac{j}{36.64 \cdot 3}\right],\ \mathrm{m/s}$$

$$= 0.804 + 0.073j\ \mathrm{mm/s}$$

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$|\hat{\vec{u}}| = 0.807 \ \mathrm{mm/sec}$; $\arg \hat{\vec{u}} = 5.189°$

$u(r,t) = |\hat{\vec{u}}| \cos(\omega t - kr + \arg \hat{\vec{u}}) \ \mathrm{m/s}$

$u(r,t) = 0.807 \cos(2\pi \cdot 2000t - 36.64 \cdot 3 + 5.189°) \ \mathrm{mm/s}$

Remark $|\hat{\vec{u}}| = \left| \frac{\hat{A}/r}{\rho_0 c}\left[1 - \frac{i}{kr}\right] \right|$

$= \frac{1}{\rho_0 c} |\hat{A}| \left| 1 - \frac{i}{kr} \right|$

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Acoustic displacement for harmonic spherical waves

Given $\hat{u}(r,t) = \hat{U} e^{j(\omega t - kr)}$; $\hat{\xi}(r,t) = \hat{X} e^{j(\omega t - kr)}$

$\Rightarrow u = \frac{d\xi}{dt}$; $\hat{U} \cancel{e^{j(\omega t - kr)}} = j\omega \hat{X} \cancel{e^{j(\omega t - kr)}}$

$$\hat{X} = \frac{\hat{U}}{j\omega} \Rightarrow |\hat{X}| = \frac{|\hat{U}|}{|j|\,\omega} = \frac{|\hat{U}|}{\omega}$$

Continuing numerical example:

$\hat{U} = 0.804 + 0.073j$ mm/s ; $\omega = 2\pi f = 2\pi \cdot 2000$

$$\hat{X} = 0.064\, e^{-j85^\circ} = \frac{0.804 + 0.073j}{j \cdot 2 \cdot \pi \cdot 2000} = ??\ \mu m$$

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$$\xi(r,t) = \mathrm{Re}\left[\hat{\xi}(r,t)\right] = 0.064\cos(2\pi\cdot 2000t - 36.64\cdot 3 - 85^\circ)\ \mu m$$

$\uparrow$ 64 nm!!

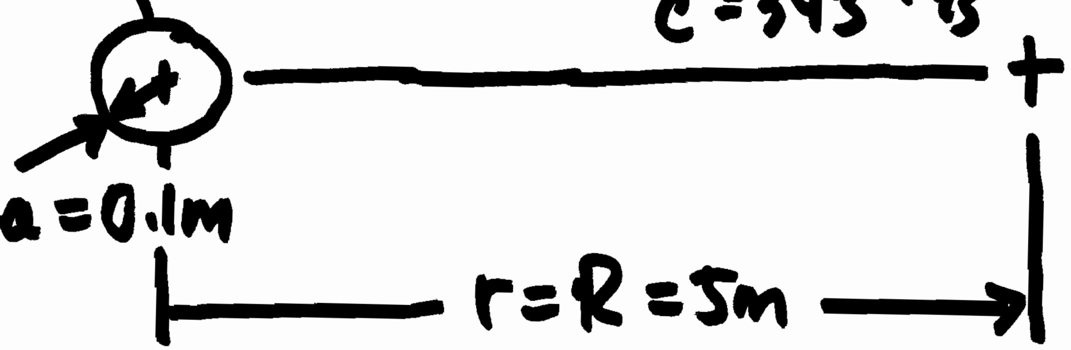
Practical Situation

what vibration amplitude rqd??

$\rho_0 = 1.21\ kg/m^3$

$c = 343\ m/s$

measure pressure amplitude of 5Pa



$f = 1000\ Hz$