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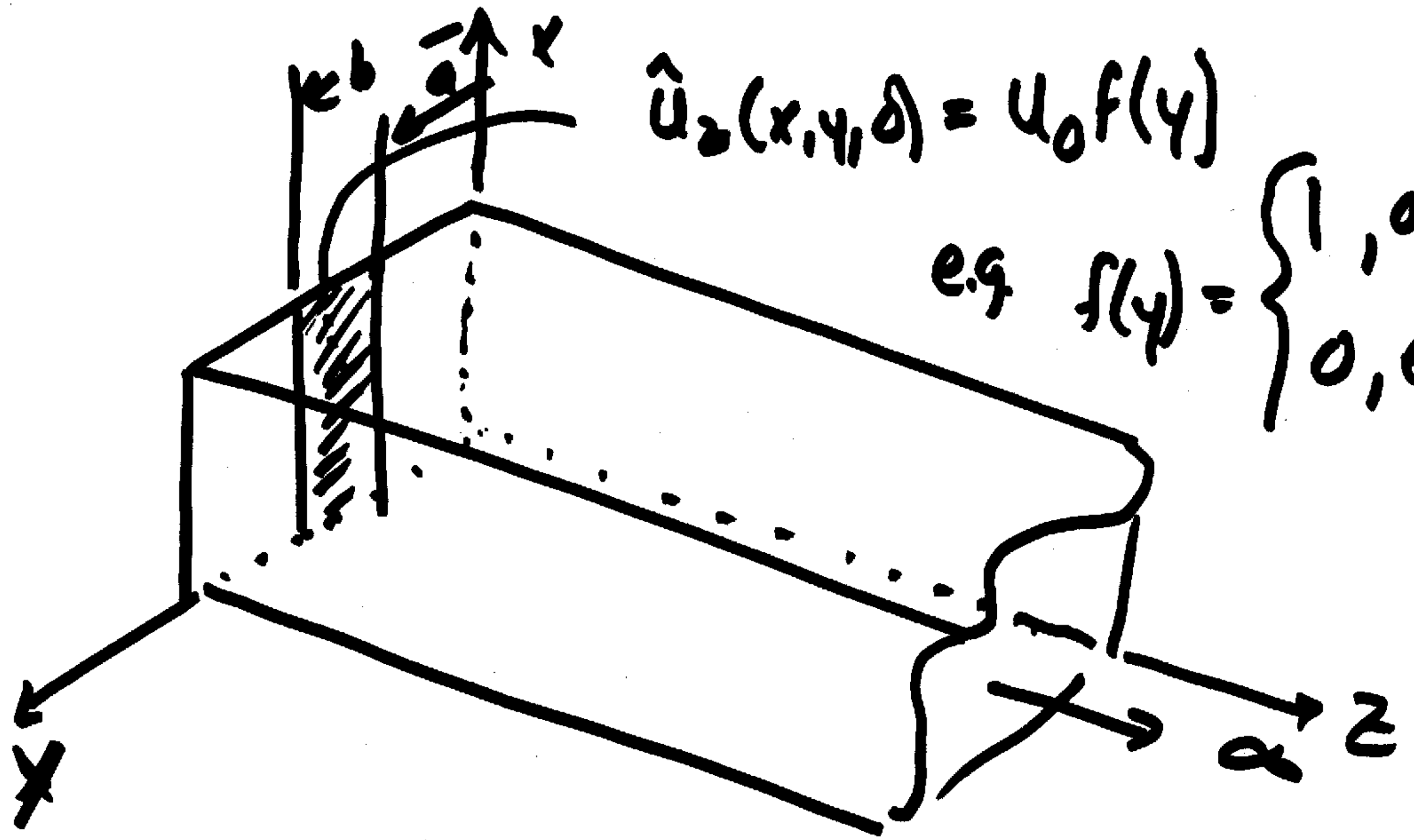
Consider the following problem

(2D Rectangular Waveguide)



$$\hat{u}_z(x,y,0) = U_0 f(y)$$

e.g. $f(y) = \begin{cases} 1, & a < y < b \\ 0, & \text{elsewhere} \end{cases}$

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Then, it is reasonable to expect that the acoustic pressure will not depend on the x coordinate;

$e^{j\omega t}$

$$\hat{p}(x,y,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \hat{A}_{lm} \cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)e^{j(\omega t - k_z z)}$$

For this problem, only include $l=0$, and the sum reduces to

$e^{j\omega t}$

$$\hat{p}(y,z) = \sum_{m=0}^{\infty} \hat{A}_{0m}\cos\left(\frac{m\pi}{L_y}y\right)e^{j(\omega t - k_z z)}$$

$$k_z = \pm\sqrt{\hat{k}^2 - \left(\frac{m\pi}{L_y}\right)^2}\ , \quad \hat{k} = \frac{\omega}{c} - j\alpha$$

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$$\hat{u}_z(y,z) = \cancel{-}\frac{1}{\rho_0 \cancel{j}\omega} \sum_{m=0}^{\infty} (\cancel{-j}k_z)\hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) e^{-jk_z z}$$

At $z=0$;

$$U_0 f(y) = \sum_{m=0}^{\infty} \frac{k_z}{\rho_0 \omega} \hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) \cancel{e}^{\,1}$$

Use the orthogonality of $\cos\left(\frac{m\pi}{L_y} y\right)$

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$$\int_0^{L_y} \cos\left(\frac{m\pi}{L_y}y\right)\cos\left(\frac{q\pi}{L_y}y\right)dy = \begin{cases} \frac{L_y}{2} ; & m = q \\ 0 ; & m \neq q \end{cases}$$

Then

$$\hat{A}_{om} = \cancel{\frac{\rho_0 \omega U_0}{k_z}\frac{2}{L_y}\int_0^{L_y} \cos\left(\frac{m\pi}{L_y}y\right)} f(y)\,dy$$

$$\hat{A}_{om} = \begin{cases} \dfrac{\rho_0 \omega U_0}{k_z}\dfrac{1}{L_y}\displaystyle\int_0^{L_y} f(y)\,dy, & m = 0 \\[2ex] \dfrac{\rho_0 \omega U_0}{k_z}\dfrac{2}{L_y}\displaystyle\int_0^{L_y} \cos\left(\frac{m\pi}{L_y}y\right) f(y)\,dy, & m > 0 \end{cases}$$

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So now, we know the acoustic pressure

$$\hat{p}(x,z)e^{j\omega t} = \sum_{m=0}^{\infty} \hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) e^{j(\omega t - k_z z)}$$

$$k_z = \pm\sqrt{\hat{k}^2 - \left(\frac{m\pi}{L_y}\right)^2}$$

Axial wavenumber, let $\hat{k} = \frac{\omega}{c}$, $\alpha = 0$

$$k_z = \begin{cases} +\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{L_y}\right)^2} \;; & \frac{\omega}{c} > \frac{m\pi}{L_y} \;; \text{ Propagating} \\ \pm j\sqrt{\left(\frac{m\pi}{L_y}\right)^2 - \left(\frac{\omega}{c}\right)^2} \;; & \frac{\omega}{c} < \frac{m\pi}{L_y} \;: \text{ Evanescent} \end{cases}$$

choose this root for decaying amplitude.

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For a fixed frequency ω; define

$$M = \mathrm{INT}\left(\frac{L_y \omega}{\pi c}\right)^2 \leftarrow ??$$

MathCAD INT = floor

Then, the acoustic pressure can be split into two parts

$$p(y,z)e^{j\omega t} = \sum_{m=0}^{M} \hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) e^{j(\omega t - k_z z)}$$

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{L_y}\right)^2}$$

propagating

$$+ \sum_{m=M+1}^{\infty} \hat{A}_0 \cos\left(\frac{m\pi}{L_y} y\right) e^{-\gamma z} e^{j\omega t}$$

evanescent

$$\gamma = \sqrt{\left(\frac{m\pi}{L_y}\right)^2 - \left(\frac{\omega}{c}\right)^2}$$

University of Idaho The <u>highest</u> frequency at which <u>only</u> the plane mode will propagate can be found from

$$\frac{\omega_c}{c} = \frac{1 \cdot \pi}{L_y} \qquad \omega_c = \text{Cutoff Frequency}$$

$$\omega_c = \frac{\pi}{L_y} c \Rightarrow 2\cancel{\pi} f_c = \frac{\cancel{\pi}}{L_y} c$$

$$\boxed{f_c = \frac{1c}{2L_y}}$$

$$\boxed{L_y = \frac{1}{2}\frac{c}{f_c} = \frac{1}{2}\lambda_c}$$

Upper limit on waveguide size for which only plane mode will propagate at a freq f_c.