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Phase Speed and Dispersion

$$\hat{p}(y,z)e^{j\omega t} = \sum_{m=0}^{\infty} \hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) e^{j(\omega t - k_z z)}$$

$$k_z = \begin{cases} \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{L_y}\right)^2}, & \frac{\omega}{c} > \frac{m\pi}{L_y}, \text{ prop} \\ -j\sqrt{\left(\frac{m\pi}{L_y}\right)^2 - \left(\frac{\omega}{c}\right)^2}, & \frac{\omega}{c} < \frac{m\pi}{L_y}, \text{ evanescent} \end{cases}$$

Apply the an identity to $\cos\left(\frac{m\pi}{L_y} y\right)$ to obtain:

$$\hat{p}(y,z)e^{j\omega t} = \sum_{m=0}^{\infty} \hat{A}_{0m} \frac{1}{2}\left[e^{-j\frac{m\pi}{L_y}y} + e^{j\frac{m\pi}{L_y}y}\right] e^{j(\omega t - k_z z)}$$

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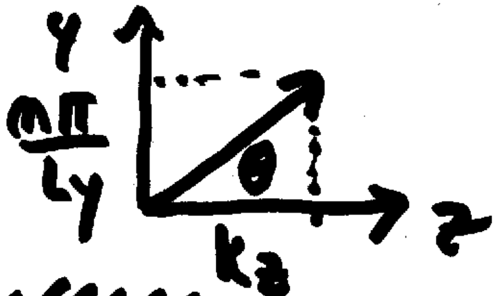
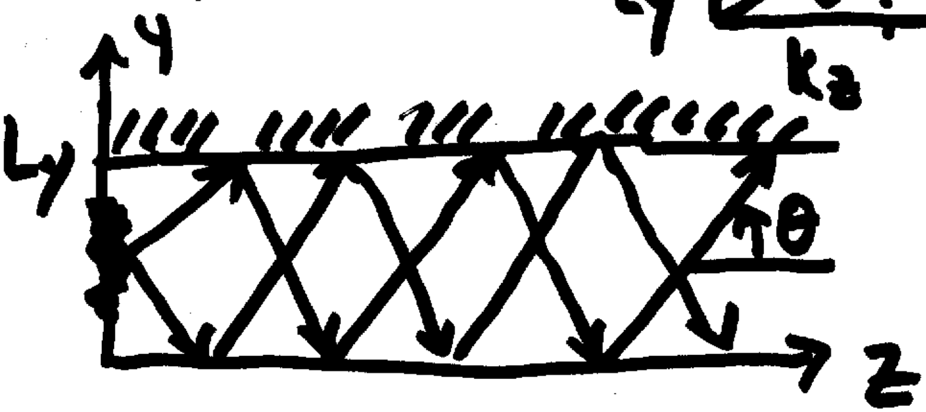
Or; For propagating modes

$$\hat{p}(x,y,z)e^{j\omega t} = \sum_{m=0}^{\infty} \frac{\hat{A}_{0m}}{2} e^{j\left(\omega t - k_z z - \frac{m\pi}{L_y} y\right)} + \frac{\hat{A}_{0m}}{2} e^{j\left(\omega t - k_z z + \frac{m\pi}{L_y} y\right)}$$



$$\cos\theta = \frac{k_z}{\sqrt{k_z^2 + \left(\frac{m\pi}{L_y}\right)^2}}$$

$m=1$ comp



$$\cos\theta = \sqrt{1 - \left(\frac{m\pi c}{\omega L_y}\right)^2}$$

When $\theta > 90^\circ$, modes become evanescent

University of Idaho Dispersion, write the pressure again as:

$$p(y,z)e^{j\omega t} = \sum_{m=0}^{\infty} \hat{A}_{0m} \cos\left(\frac{m\pi}{L_y} y\right) e^{jk_z\left(\frac{\omega}{k_z} t - z\right)}$$

$$c_p = \frac{\omega}{k_z} \equiv \text{phase speed}$$

$$c_p = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1-\left(\frac{m\pi c}{\omega L_y}\right)^2}}$$ ⇒ phase speed c_p decrease with increasing m.

Alternately, c_p is a function of frequency ⇒ Dispersion.

University of Idaho **Three-Dimensional Semi-Infinite Rectangular Wave Guide**

$$\hat{p}(x,y,z)e^{j\omega t} = \sum_{m=0}^{\infty} \hat{A}_{lm} \cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)e^{j(\omega t - k_z z)}$$

and $k_z = \begin{cases} \sqrt{\left(\frac{\omega}{c}\right.} \\ \\ \end{cases}$

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$$\hat{p}(x,y,z)e^{j\omega t} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \hat{A}_{lm}\cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)e^{j(\omega t - k_z z)}$$

$$k_z = \begin{cases} \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{l\pi}{L_x}\right)^2 - \left(\frac{m\pi}{L_y}\right)^2}, & \left(\frac{\omega}{c}\right)^2 > \left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \\ -j\sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 - \left(\frac{\omega}{c}\right)^2}, & \left(\frac{\omega}{c}\right)^2 < \left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \end{cases}$$

BC: $\hat{u}_z(x,y,z=0)e^{j\omega t} = U_0 f(x,y)e^{j\omega t}$

$e^{-jk_z z}$

$$\hat{u}_z(x,y,z) = -\frac{1}{\rho_0 j\omega}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} -jk_z\hat{A}_{lm}\cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)$$

University of Idaho at $z=0$

$$U_0 f(x,y) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{k_z \hat{A}_{lm}}{\rho_0 \omega} \cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)$$

<u>Exploit orthogonality</u> ~~cof~~ $\int^{L_x} \cos\left(\frac{l\pi}{L_x}x\right)\cos\left(\frac{q\pi}{L_x}x\right)dx$

$$\hat{A}_{lm} = \begin{cases} \frac{\rho_0 \omega U_0}{k_z}\frac{1}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\,dx\,dy, & l=m=0 \\ \frac{\rho_0 \omega U_0}{k_z}\frac{2}{L_x L_y}\int_0^{L_x}\int_0^{L_y} \cos\left(\frac{m\pi}{L_y}y\right) f(x,y)\,dx\,dy, & l=0,\ m>0 \\ \frac{\rho_0 \omega U_0}{k_z}\frac{2}{L_x L_y}\int_0^{L_x}\int_0^{L_y} \cos\left(\frac{l\pi}{L_x}x\right) f(x,y)\,dx\,dy, & l>0,\ m=0 \end{cases}$$

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$$\frac{\rho_0 \omega U_0}{k_z} \frac{4}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \cos\left(\frac{l\pi}{L_x} x\right) \cos\left(\frac{m\pi}{L_y} y\right) f(x,y)\, dx\, dy$$

$$m > 0, \; l > 0$$

$$\cos\theta = \sqrt{1 - \left(\frac{l\pi c}{\omega L_x}\right)^2 - \left(\frac{m\pi c}{\omega L_y}\right)^2}$$ = direction cosine between propagating mode and z axis

$$c_p = \frac{c}{\cos\theta}$$ = phase speed of lm mode