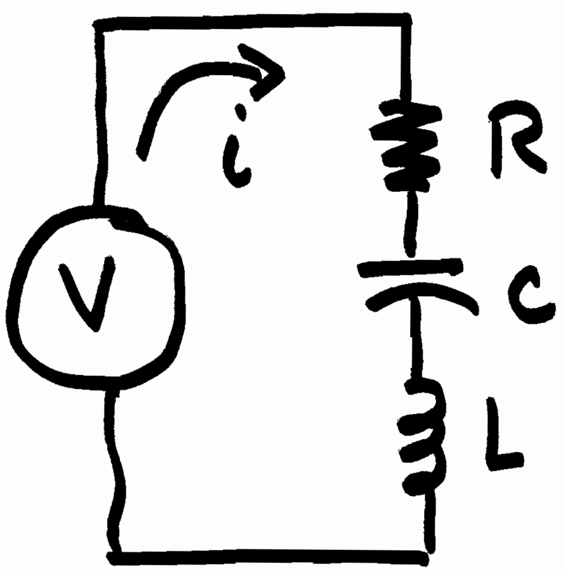
Application of Complex Variables to Analyze Electrical Systems



Restrict ourselves to harmonic inputs:

$$V(t) = V\cos\omega t = V\cos 2\pi f t$$

We seek the steady state current i.

$$V_R = Ri, \quad V_C = \frac{1}{C}\int i\,dt, \quad V_L = L\frac{di}{dt}$$

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$$V(t) = Ri + \frac{1}{C}\int i\,dt + L\frac{di}{dt}$$

We can use complex variables to solve for $i(t)$;

$V(t) \rightarrow \hat{V}e^{j\omega t}$, $\hat{V} = V + 0j$, $i(t) = \hat{I}e^{j\omega t}$

$$\hat{V} = R\hat{I} + \frac{1}{j\omega C}\hat{I} + Lj\omega\hat{I}$$

solve for $\hat{I}$;

$$\hat{I} = \frac{1}{R + \frac{1}{j\omega C} + j\omega L}\hat{V}$$

Given $V, R, C, L, \omega \Rightarrow \bar{i}(t) = |\hat{I}|\cos(\omega t + \arg\hat{I})$

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Like the mechanical system, we can define an impedance for an electrical circuit:

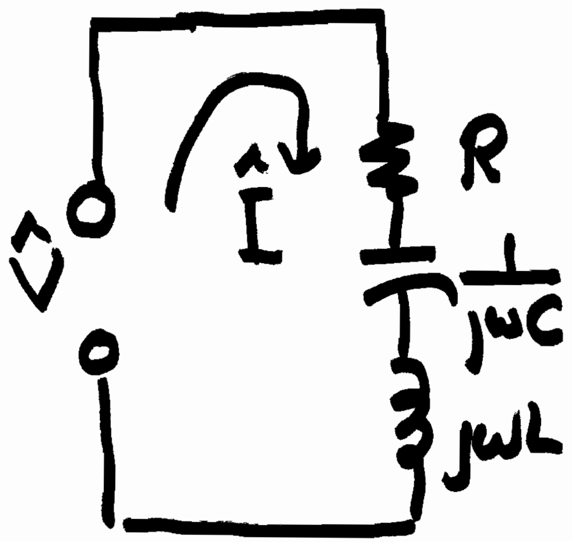
The electrical mi impedance for a circuit is

$$\hat{Z}_e = \frac{\hat{V}}{\hat{I}} \Rightarrow \hat{V} = \hat{Z}_e \hat{I}$$

For the series RLC circuit; the electrical impedance is:

$$\hat{Z}_e = R + \frac{1}{j\omega C} + j\omega L$$

Using electrical impedances to solve circuit problems



Resistor $V_R = Ri$; $\hat{V}_R = R\hat{I}$

Capacitor $V_C = \frac{1}{C}\int i\,dt$; $\hat{V}_C = \frac{1}{j\omega C}\hat{I}$

Inductor $V_L = L\frac{di}{dt}$; $\hat{V}_L = j\omega L\hat{I}$

Using "DC" analysis $\hat{Z}_e = R + \frac{1}{j\omega C} + j\omega L$

$$\hat{V} = \left(R + \frac{1}{j\omega C} + j\omega L\right)\hat{I}$$

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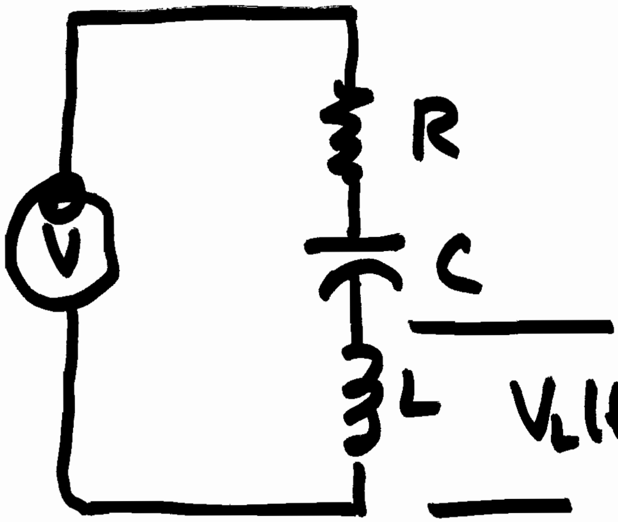
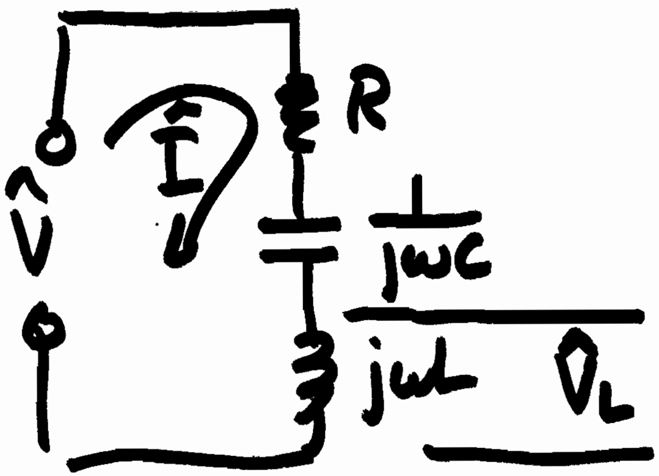
Examples:

$v(t) = V\cos \omega t$



$v_L(t) = ?$



$$\hat{V}_L = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}} \hat{V}$$

$$v_L(t) = |\hat{V}_L| \cos(\omega t + \arg \hat{V}_L)$$

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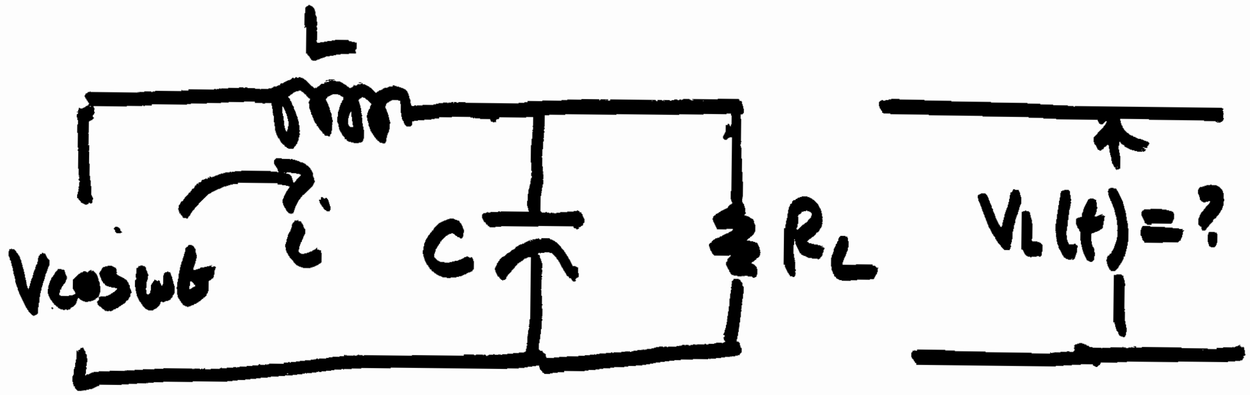
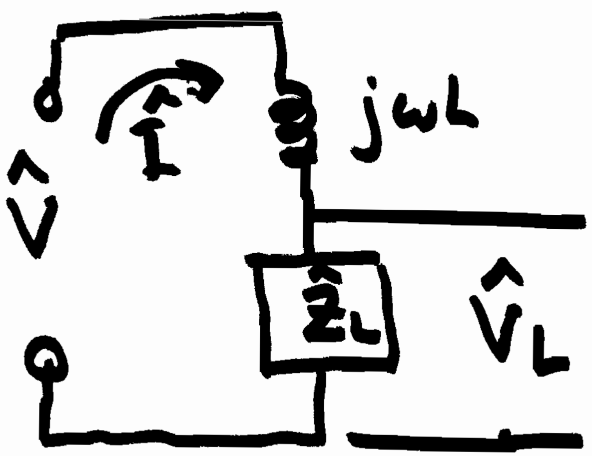




$$\hat{Z}_L = \frac{1}{\frac{1}{R_L} + j\omega C}$$

$$\hat{V}_L = \frac{\hat{Z}_L}{\hat{Z}_L + j\omega L}\hat{V}$$

$$\hat{V} = (j\omega L + \hat{Z}_L)\hat{I} \Rightarrow \hat{I} = \frac{1}{j\omega L + \hat{Z}_L}\hat{V}$$

$$i(t) = |\hat{I}|\cos(\omega t + \arg \hat{I}).$$

Analogies Between Electrical and Mechanical Systems

Mechanical System (mass-spring-damper)

Electrical System (series-RLC)

$$F\cos\omega t = s\int u\,dt + R_m u + m\dot{u}$$

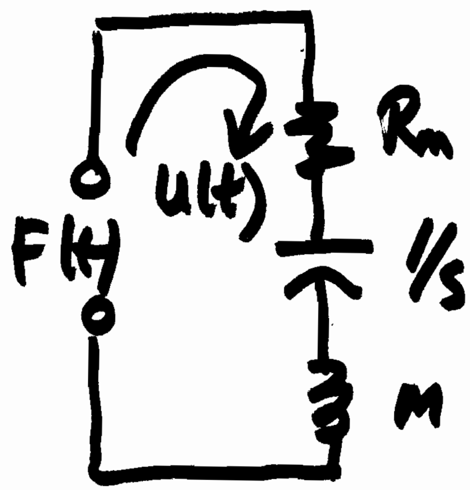
$$V\cos\omega t = \frac{1}{C}\int i\,dt + Ri + L\frac{di}{dt}$$

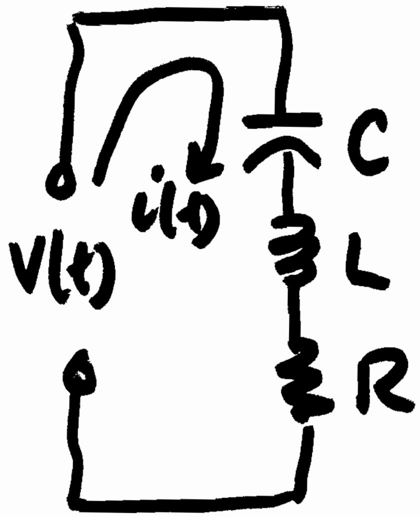
These systems are analogous because they share the same differential equation

In this case $F(t) \leftrightarrow V(t)$, $i(t) \leftrightarrow u(t)$

$m \leftrightarrow L$, $R \leftrightarrow R_m$, $1/s \leftrightarrow C$

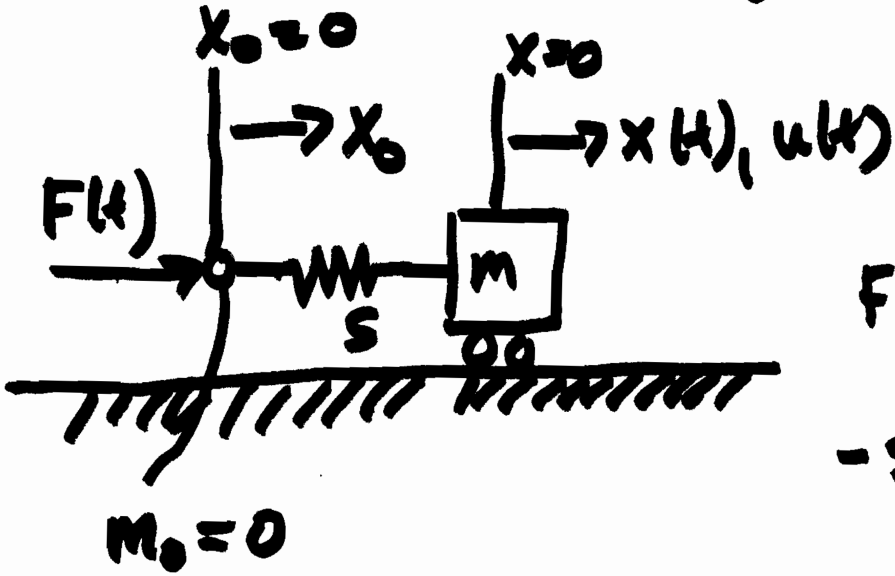
We can draw equivalent circuit diagrams...





Problems in text, given a mechanical system, and asked to find a circuit analog.

Newton's laws for the following mechanical system...



$$F(t) - s(x_0 - x) = \cancelto{0}{m_0}\ddot{x}_0 = 0$$

$$-s(x - x_0) = m\ddot{x}$$