From the state variable chain, identify relationships between the derivatives of the state variables:

\[ x_1 = x_2 \]
\[ x_4 = x_2 \]
\[ x_3 = x_3 \]

Substitute these variable relationships into the system model:

\[ 5x_2 + 12x_2 + 5x_1 - 8x_4 - 4x_3 = 0 \]
\[ 3x_4 + 8x_4 + 1x_3 - 8x_2 - 4x_1 = f(x) \]

Solve the time derivative:

\[ x_2 = \frac{2}{3} x_1 + \frac{8}{3} x_2 - \frac{4}{3} x_3 - \frac{8}{3} x_4 + \frac{1}{2} f(x) \]

\[ x_0 = \frac{4}{3} x_1 + \frac{8}{3} x_2 - \frac{4}{3} x_3 - \frac{8}{3} x_4 + \frac{1}{2} f(x) \]
Internal Variables and Kinematic Constraints: Dependent variables and model equations w/o time-derivatives

Precise example

\[ I_1, r_1, \theta_1 \]

\[ k_4 \]

\[ c_2, \phi \]

\[ I_2, r_2 \]

\[ \theta_2 \]

\[ \phi(t) \text{ specified angular position of end of shaft } k_4 \]

System Model

\[ -k_4 (\theta_1 - \phi) - F r_1 = I_1 \ddot{\theta}_1 \]

\[ -c_2 \dot{\theta}_2 + F r_2 = I_2 \ddot{\theta}_2 \]

\[ r_1 \dot{\theta}_1 = r_2 \dot{\theta}_2 \]

Unknowns: \( \theta_1, F, \theta_2 \)
If

1. There are any dependent variables in the system model, algebraically eliminate them first.

2. If any system model equations do not contain time derivatives, algebraically eliminate them.

Solve for \( F \) in the 1st put into the second:

\[-c_{T} \dot{\Theta}_{2} - \frac{r_{2}}{r_{1}} k_{T}(\Theta_{1} - \Theta) = \frac{r_{2}}{r_{1}} I_{1} \ddot{\Theta}_{1} = I_{2} \ddot{\Theta}_{2}\]

\[r_{1} \dot{\Theta}_{1} = r_{2} \dot{\Theta}_{2}\]

Eliminate the second equation by solving for \( \Theta_{1} \) in the first, and place into the 1st:

\[\frac{r_{2}}{r_{1}} k_{T} \dot{\Theta} = \left(\frac{r_{2}}{r_{1}}\right)^{2} k_{T} \dot{\Theta}_{2} + c_{T} \dot{\Theta}_{2} + \left[I_{2} + \left(\frac{r_{2}}{r_{1}}\right)^{2} I_{1}\right] \ddot{\Theta}_{2}\]
Derivatives chain:

\[ \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3 \]

State variable assignment:

\[ x_1 = \theta_2, \quad x_2 = \theta_3 \]

Relationship between the derivative of state var:

\[ \frac{R^2}{n_1} \dot{x}_1 = \frac{1}{n_2} \left[ \frac{(\frac{L_2}{\gamma})^2}{T_2} + \left( \frac{L_2}{\gamma} \right)^2 \right] x_2 \]

Substitute state var into system model eqns:

\[ \frac{(n_2/\gamma)^2}{T_2} \frac{R^2}{n_1} \dot{x}_1 = \frac{(n_2/\gamma)^2}{T_2} \frac{R^2}{n_1} \dot{x}_1 \]

Isolate third derivative:

\[ x_2 = -\frac{(n_2/\gamma)^2}{T_2} \frac{R^2}{n_1} \frac{(n_2/\gamma)^2}{T_2} \frac{R^2}{n_1} \dot{x}_1 \]
Numerical Solution of Differential Equations

Simulation = solution

Let's start with a simple example:

\[ \dot{x} + ax = u(t), \quad x(0) = 0 \]

\[ u(t) \]

---

Euler method of solution

Isolate the time derivative:

\[ \dot{x} = -ax + u(t) \]

Approximate the time derivative:

\[ \dot{x} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} \]
\[
\frac{x(t+\Delta t) - x(t)}{\Delta t} = -ax(t) + u(t)
\]

Solve for \(x(t+\Delta t)\)

\[
x(t+\Delta t) = \left[-ax(t) + u(t)\right] \Delta t + x(t)
\]

To solve, we generate the following table

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x(t))</th>
<th>(\Delta t \left[-ax(t) + u(t)\right] + x(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(0.101\left[-1\cdot0 + 1\right] + 0 = 0.101)</td>
</tr>
<tr>
<td>(\Delta t)</td>
<td>0.101</td>
<td>(0.101\left[-1\cdot0.101 + 1\right] + 0.101 = 0.1918)</td>
</tr>
<tr>
<td>(2\Delta t)</td>
<td>0.202</td>
<td>(0.101\left[-1\cdot0.1918 + 1\right] + 0.1918 = 0.2735)</td>
</tr>
<tr>
<td>(3\Delta t)</td>
<td>0.303</td>
<td>0.2935</td>
</tr>
</tbody>
</table>